A MINIMAL PRIME MODEL WITH AN INFINITE SET OF INDISCERNIBLES

BY

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ABSTRACT

We show the existence of a prime minimal model containing an infinite set of indiscernibles. We then find a sentence of $L_{\omega_1\omega}$ which is categorical in ω_1 but whose model of power ω_1 is not $(\omega_1, L_{\omega_1\omega})$-homogeneous. This answers a question posed by Keisler.

1. The purpose of this paper is to answer negatively the following question of Keisler from [2] page 101*: Let T be a countable set of sentences of $L_{\omega_1\omega}$ which is ω_1-categorical. Is the model of T of power ω_1 $(\omega_1, L_{\omega_1\omega})$-homogeneous? The crucial result is the following theorem, which answers a question asked by Victor Harnik.

THEOREM. *There exists a countable first order theory having a prime minimal model which contains an infinite set of indiscernibles.*

For other examples of minimal models, see Fuhrken [1].

The author is indebted to Saharon Shelah for pointing out that the above theorem can be used to answer Keisler's question and for many other valuable suggestions. Let κ be an infinite cardinal, M, N models, and $|M|$, $|N|$ their universe sets. A theory T is *κ-categorical* if every two models of T of power κ are isomorphic. If $A \subset |M|$ and f is a function from A into $|N|$ then f is said to be an *$L_{\omega_1\omega}$-elementary mapping* from (A,M) into $|N|$ if for every formula $\phi(x_1,\cdots,x_n) \in L_{\omega_1\omega}$ and for any $a_1,\cdots,a_n \in A$, $M \models \phi[a_1,\cdots,a_n]$ iff $N \models \phi[f(a_1),\cdots,f(a_n)]$. A model M is *$(\kappa, L_{\omega_1\omega})$-homogeneous* if for every $A \subset |M|$ of power $< \kappa$ and $b \in |M|$ every $L_{\omega_1\omega}$-elementary mapping from (A,M) into $|M|$ can be extended to an $L_{\omega_1\omega}$-elementary mapping of $A \cup \{b\}$ into $|M|$. M is a

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* The same example of our section 3 also answers the questions stated by Keisler on page 100 and in remark 3 on page 123 of [2].

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prime model if $N \equiv M \Rightarrow M$ is isomorphic to a (first order) elementary submodel of N. The first order formula $\phi(x_1, \cdots, x_n)$ *isolates* the type of $\langle a_1, \cdots, a_n \rangle$ in M if $M \models \phi[a_1, \cdots, a_n]$ and for all $a'_1, \cdots, a'_n \in |M|$, $M \models \phi[a'_1, \cdots, a'_n] \Rightarrow \langle a'_1, \cdots, a'_n \rangle$ and $\langle a'_1, \cdots, a'_n \rangle$ realize the same type in M. M is *atomic* if every finite sequence in M realizes an isolated type in M.

For results on prime models, especially that M is prime iff M is countable and atomic, see [3]. M is *minimal* if $N \prec M \Rightarrow N$ is M. $A \subset |M|$ is a *set of indiscernibles* if for every distinct $a_1, \cdots, a_n \in A$ and distinct $a'_1, \cdots, a'_n \in A$ and first order formula $\phi(x_1, \cdots, x_n)$, $M \models \phi[a_1, \cdots, a_n]$ iff $M \models \phi[a'_1, \cdots a'_n]$. For definition of other concepts consult the references.

2. Proof of the theorem

We define inductively a sequence of sets $\{B_i: i < \omega\}$ and a sequence of first order languages $\{L_i: i < \omega\}$ such that:

a) B_0 is an infinite set, L_0 is $\{=, P_0(x)\}$. (B_0 will turn out to be the set of indiscernibles). $P_0(x)$ iff $x \in B_0$.

b) For all $b \in B_{i+1}$, $c \in B_i$ there is a predicate $\theta(x_1, x_2) \in L_{i+1}$ such that $\theta(b, x_2)$ iff $x_2 = c$. (Every element of B_{i+1} will define every element of B_i).

c) Let H be the set of permutations of B_0 which move only a finite number of elements. For all $f \in H$ and $i < \omega$ there is a unique extension of f to an automorphism $f^{(i)}$ of $\cup_{j \leq i} B_j$. Let $H^{(i)} = \{f^{(i)}: f \in H\}$.

d) For all $i \in \omega$ there is a predicate $P_i(x) \in L_i$ such that $P_i(x)$ iff $x \in B_i$.

e) For all $0 < i$, $n < \omega$, $b_1, \cdots, b_n \in \cup_{j \leq i} B_j$ there is a predicate $R(x_1, \cdots, x_n) \in L_i$ such that $R(b'_i, \cdots, b'_n)$ iff there is an automorphism $f \in H^{(i)}$ of $\cup_{j \leq i} B_j$, $f(b_l) = b'_l$ $(1 \leq l \leq n)$. (The $R(\cdots)$'s will be names for the types realized in $\cup_{i<\omega} B_i$).

f) B_i and L_i are countable.

The definition proceeds as follows: B_0, L_0 are already defined and obviously satisfy the conditions. Now assume that $\cup_{j \leq i} B_j$, $\cup_{j \leq i} L_j$ are defined as required. Let b be a new element and for every $c \in B_i$ let $\theta_{bc}(x_1, x_2)$ be a new predicate symbol. Define $\theta_{bc}(b, x_2)$ iff $x_2 = c$. Let $H^{(i)} = \{f_n: n < \omega\}$ be the set of automorphisms we must extend to B_{i+1}. (There are just $\aleph_0$ because of the uniqueness condition in c)). Clearly H is a group under composition. By the uniqueness condition, $H^{(i)}$ is also. Let $B_{i+1} = \{b_f: f \in H^{(i)}\}$ and define the relations $\theta_{bc} = \{\langle b_f, f(c) \rangle : f \in H^{(i)}\}$. Then B_{i+1} satisfies *c*) where the unique extension of $g \in H^{(i)}$ to B_{i+1} is given by $g(b_f) = b_{g \cdot f}$, and it is easily seen that *b*) is satisfied.

Let $P_{i+1}(x)$ be a new predicate symbol such that $P_{i+1}(x)$ iff $x \in B_{i+1}$. Now define relations $R(x_1, \cdots, x_n)$ on $\cup_{j<i+1} B_j$ as required by *e*). Let $L_{i+1} = \{\theta_{bc}(x_1, x_2): c \in B_i\} \cup \{R_{\bar{b}}(x_1, \cdots, x_n): n < \omega, \quad \bar{b} = \langle b_1, \cdots, b_n \rangle, \quad b_k \in \cup_{j \leq i+1} B_j\} \cup \{P_{i+1}(x)\}$.

Let $M = \langle \cup_{i<\omega} B_i, \cup_{i \leq \omega} L_i \rangle$.

M is prime: Let $b_1, \cdots, b_n \in |M|$. Assume $b_1, \cdots, b_n \in \cup_{j \leq i} B_j$. Let $R(x_1, \cdots, x_n) \in L_i$ be the associated predicate from e). If $R(b'_1, \cdots, b'_n)$ then $b'_1, \cdots, b'_n \in \cup_{j \leq i} B_j$ and there is an automorphism $f \in H^{(i)}$ which takes b_l to $b'_l (1 \leq l \leq n)$. f may be extended to $\cup_{i<\omega} B_i$ and so $\langle b_1, \cdots, b_n \rangle$ and $\langle b'_1, \cdots, b'_n \rangle$ realize the same type. In other words $R_{\bar{b}}(x_1, \cdots, x_n)$ isolates the type of $\langle b_1, \cdots, b_n \rangle$. So M is atomic. As M is obviously countable, M is prime.

M is minimal: Let $c \in M$, $N \prec M$. We must show $c \in N$. Assume $c \in B_i$. $N \models (\exists x) P_{i+1}(x)$; if b is such an x, then there is $\theta(x_1, x_2)$ such that $M \models (\exists ! \, x_2) \theta(b, x_2) \wedge \theta(b, c)$. So c must be in N.

$A = B_0$ is a set of indiscernibles: By c) for every distinct $a_1, \cdots, a_n \in A$ and distinct $a'_1, \cdots, a'_n \in A$ the map $a_i \to a'_i$ may be extended to a permutation in H which in turn may be extended to an automorphism of $\cup_{i<\omega} B_i$. So every two finite sequences of the same length from A realize the same type.

Thus M is the required model and the proof is complete.

3. Now we answer Keisler's questions. Let $\psi \in L_{\omega_1 \omega}$ be a Scott sentence of M (see [2]). Let T be a theory in $L_{\omega_1 \omega}$ which has one equivalence relation for which each equivalence class is a model of ψ. In particular each equivalence class is atomic, and since M is minimal each equivalence class is countable.* Hence each equivalence class is isomorphic to M, and thus T is categorical in every uncountable power. So N, T's model in ω_1, is made up of ω_1 equivalence classes $\{M_i: i < \omega_1\}$ each isomorphic to M. Let A be the set of indiscernibles in M_0 and let $a \in A$. If f is a $1-1$ map of $A - \{a\}$ onto A then f is $L_{\omega_1 \omega}$-elementary since for any $a_1, \cdots, a_n \in A - \{a\}$ there is an automorphism of M_0 taking a_l onto $f(a_l)$ $(1 \leq l \leq n)$. But f cannot be extended to an $L_{\omega_1 \omega}$-elementary mapping of A into $|N|$. So N is not $(\omega_1, L_{\omega_1 \omega})$-homogeneous.

4. The following theorem of Shelah gives a positive answer to a weaker version of Keisler's question.

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* For if M_1 is an uncountable equivalence class then there are $M_2 \underset{\#}{\prec} M_3 \prec M_1$ where M_2 and M_3 are countable; M_3 is isomorphic to M, contradicting M's minimality.

THEOREM (Shelah) *Let $\Phi \in L_{\omega_1\omega}$ be an ω_1-categorical sentence having models of arbitrarily large power; $M \models \Phi$, M of power ω_1. Then there is a countable language $L' \subset L_{\omega_1\omega}$, $\Phi \in L'$, such that*

(i) *M is (ω_0, L')-homogeneous.*

(ii) *If $N_1 \prec_{L'} M$, $N_2 \prec_{L'} M$ where N_1, N_2 are countable, then every isomorphism of N_1 onto N_2 can be extended to an automorphism of M.*

5. We can also prove the categoricity of T in the following way. It can be shown that the minimality of any specific minimal model, for example our M, is expressible by a sentence of $L_{\omega_1\omega}$, and since ψ is a complete theory in $L_{\omega_1\omega}$, every model of ψ is minimal, hence countable. Thus T's categoricity follows just from M's minimality.

In order to get that the map f in 3) is $L_{\omega_1\omega}$-elementary we just need that there is an automorphism of M_0 taking a_l onto $f(a_l)$. This follows from M's being (first order) ω-homogeneous. In other words the whole construction in 3) will work for any homogeneous minimal model with an infinite set of indiscernibles.

REFERENCES

1.* G. Fuhrken, *Minimal-und Primmodelle*, Arch. Math. Logik Grundlagenforsch. 9 (1963), 3–11.

2. H. J. Keisler, *Model Theory for Infinitary Logic*, North-Holland, 1971.

3. R. Vaught, Denumerable models of complete theories, *Infinitistic Methods*, Pergamon Press, New York, 1961, 303–321.

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* There is an error on p. 7 where it is claimed that T has exactly three non-isomorphic minimal models. T in fact has $2^{\aleph_0}$.